

NONSINGULARITY OF THE SAMPLE COVARIANCE MATRIX

by

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Introduction.

Let $\underline{X}_1, \dots, \underline{X}_N$ be $p \times 1$ random vectors. Define $A = \sum_{\alpha=1}^N (\underline{X}_\alpha - \bar{\underline{X}})(\underline{X}_\alpha - \bar{\underline{X}})'$, where $\bar{\underline{X}} = \sum_{\alpha=1}^N \underline{X}_\alpha / N$. Dykstra [1] proved that A is nonsingular with probability 1 when $N > p$ and $\underline{X}_1, \dots, \underline{X}_N$ is a sample from a nonsingular p -variate normal distribution. In this note, the question of nonsingularity of A (with probability 1) is studied under a more general set-up and later a general (matrix) quadratic form in \underline{X}_α 's is studied from this viewpoint.

Results.

Note that $A = \bar{\underline{X}} C_N \bar{\underline{X}}'$, where

$$\bar{\underline{X}} = (\underline{X}_1, \dots, \underline{X}_N), \quad C_N = I_N - \frac{1}{N} J_N,$$

I_N is the $N \times N$ identity matrix and J_N is the $N \times N$ matrix with all elements equal to 1. Since $\text{rank } C_N = N - 1$ and $\text{rank}(A) \leq \text{rank}(C_N)$, A is singular if $p > N - 1$. In the sequel, we thus assume $p \leq N - 1$.

Theorem 1.

Suppose there exists a $(p-1)$ -dimensional hyperplane L in R^p given by $\underline{b}'\underline{x} = a$ such that $\text{Prob}[\underline{X}_\alpha \in L, \alpha = 1, \dots, N] > 0$ where $\underline{X}_1, \dots, \underline{X}_N$ are $p \times 1$ random vectors. Then $\text{Prob}[A \text{ is singular}] > 0$.

Proof:

$$\underline{X}_\alpha \in L, \alpha = 1, \dots, N \Rightarrow \underline{b}'A\underline{b} = a^2 \underline{e}_N' C_N \underline{e}_N = 0$$

$$\Rightarrow A \text{ is singular}$$

\underline{e}_N is the $N \times 1$ vector with all elements equal to 1.

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Introduction.

Let $\bar{X}_1, \dots, \bar{X}_W$ be $p \times 1$ random vectors. Define $A = \sum_{i=1}^W (\bar{X}_i - \bar{\bar{X}})(\bar{X}_i - \bar{\bar{X}})'$ where $\bar{\bar{X}} = \frac{1}{W} \sum_{i=1}^W \bar{X}_i$. Dykstra [1] proved that A is nonsingular with probability 1 when $W > p$ and $\bar{X}_1, \dots, \bar{X}_W$ is a sample from a nonsingular

p -variate normal distribution. In this note, the question of nonsingularity of A (with probability 1) is studied under a more general set-up and later a general (matrix) condition for the \bar{X}_i 's is studied from this viewpoint.

Results.

Note that $A = \bar{X}' G \bar{X}$, where

$$\bar{X} = (\bar{X}_1', \dots, \bar{X}_W')', \quad G = I_W - \frac{1}{W} \mathbf{1} \mathbf{1}'$$

\bar{X}_i is the $W \times W$ identity matrix and $\mathbf{1}$ is the $W \times W$ matrix with all elements equal to 1. Since $\text{rank } G = W - 1$ and $\text{rank}(\bar{X}) \leq \text{rank}(G)$, A is singular if $p > W - 1$. In the sequel, we thus assume $p \leq W - 1$.

Theorem 1.

Suppose there exists a $(p-1)$ -dimensional subspace L in R^p given by $g'x = 0$ such that $\text{Prob}\{X_i \in L, i = 1, \dots, W\} > 0$ where $\bar{X}_1, \dots, \bar{X}_W$ are $p \times 1$ random vectors. Then $\text{Prob}\{A \text{ is singular}\} > 0$.

Proof:

$$X_i \in L, i = 1, \dots, W \Rightarrow g'X_i = 0, i = 1, \dots, W \Rightarrow \bar{X}' G \bar{X} = 0$$

$\Rightarrow A$ is singular

is the $W \times 1$ vector with all elements equal to 1.

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Corollary 1.1.

$\text{Prob}(A \text{ is singular}) = 0$ implies that for every $(p-1)$ -dimensional hyperplane L , $\text{Prob}(\underline{X}_{\alpha} \in L, \alpha = 1, \dots, N) = 0$ which is equivalent to $\text{Prob}(\underline{X}_{\alpha} \in L) = 0$ when \underline{X}_{α} 's are i.i.d.

Lemma.

Let $\underline{X}_1, \dots, \underline{X}_p$ be $p \times 1$ mutually independent random vectors. Assume that for any $\alpha (\alpha = 1, \dots, p)$ and any $(p-1)$ -dimensional subspace L of R^p , $\text{Prob}(\underline{X}_{\alpha} \in L) = 0$. Then $\text{Prob}(\bar{\underline{X}}_{[p]} \text{ is singular}) = 0$ where $\bar{\underline{X}}_{[p]} = (\underline{X}_1, \dots, \underline{X}_p)$.

Proof:

We shall prove this by induction on p . Let $\underline{X}'_{\alpha} = (X_{\alpha 1}, \dots, X_{\alpha p})$ and $\underline{X}'_{\alpha[i]} = (X_{\alpha 1}, \dots, X_{\alpha i})$, $\bar{\underline{X}}_{[i]} = (X_{1[i]}, \dots, X_{i[i]})$, $i = 1, \dots, p$. The condition in the lemma implies that $\text{Prob}(\underline{X}_{\alpha[i]} \in L_i) = 0$ for any $(i-1)$ -dimensional subspace L_i of R^i $i = 1, \dots, p$. Clearly $\text{Prob}(\bar{\underline{X}}_{[1]} \text{ is singular}) = 0$. Assume, for $1 \leq r < p$, $\text{Prob}(\bar{\underline{X}}_{[r]} \text{ is singular}) = 0$. Suppose, $\text{Prob}(\bar{\underline{X}}_{[r+1]} \text{ is singular}) > 0$. Then there must exist a set of positive probability measure in the space of $\underline{X}_{\alpha[r+1]}$, $\alpha = 1, \dots, r$ for which

$$\begin{aligned} & \text{Prob}[\det(\underline{x}_{1[r+1]} \cdots \underline{x}_{r[r+1]}, \underline{x}_{r+1[r+1]}) \\ & = 0 | \underline{X}_{\alpha[r+1]} = \underline{x}_{\alpha[r+1]}, \alpha = 1, \dots, r] > 0. \end{aligned}$$

However, the above determinant may be written as $\underline{b}' \underline{X}_{r+1, [r+1]} = 0$, where \underline{b} is a function of $\underline{x}_{\alpha[r+1]}$, $\alpha = 1, \dots, r$ and the above conditional probability is the same as the unconditional probability which is zero by assumption unless $\underline{b} = \underline{0}$, and, in particular, the last component of \underline{b} is 0 which is the same as saying $\det(\underline{x}_{1[r]}, \dots, \underline{x}_{r[r]}) = 0$. But the probability of such event, by the induction hypothesis is 0. Hence $\text{Prob}(\bar{\underline{X}}_{[r]} \text{ is singular}) = 0$.

Proposition 1.1. Let \mathcal{A} be a $(2n-1)$ -dimensional algebra over \mathbb{C} . Then \mathcal{A} is singular if and only if $\dim \mathcal{A} = 2n-1$.

of the shape of which $\mathcal{H} = (H_1, \dots, H_n) = (H_1, \dots, H_n)$ consists of

$$b_{1,1} \sin \alpha' = (1 - \mu) \cos \alpha$$

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...erhöhter Gehalt: mindestens 1 x 900 I, ..., 100 I

System does not have any $(n = 1, \dots, r)$ and any $(n-1)$ -th order subspaces

if $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ and $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$, then $\text{Top}(\mathcal{C}) = \text{Top}(\mathcal{C}_1) \cup \text{Top}(\mathcal{C}_2)$.
 if $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ and $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$, then $\text{Top}(\mathcal{C}) = \text{Top}(\mathcal{C}_1) \cup \text{Top}(\mathcal{C}_2)$.

$$\cdot (\underbrace{x_1, \dots, x_n}_{\substack{\text{array} \\ \text{of } n \text{ elements}}}) = [\lg n]$$

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$(x_1, \dots, x_n) = \frac{1}{n} \sum_{i=1}^n x_i$ is the arithmetic mean of the x_i 's.

$$\cdot, \dots, \cdot = \cdot, ([\cdot]_1^K, \dots, [\cdot]_n^K) = [\cdot]_1^K, ([\cdot]_1^K, \dots, [\cdot]_n^K) = [\cdot]_1^K \text{ bzw.}$$

The condition in the lemma implies that $\text{Tr}(\rho_{A_i}^{(k)}) = 0$ for any

classical $\mathcal{G}_1, \dots, \mathcal{G}_n = \mathcal{G}$ is \mathcal{G} is a quadratic Lie algebra, (1-1)

Prop(\bar{X}_{-1}) is singular) = 0. Assume, for $1 \leq i \leq p$, Prop(\bar{X}_{-i}) is singular) = 0.

Suppose, $\text{Prob}\left(\frac{\bar{X}}{\sqrt{\frac{1}{n}}}\right)$ is singular > 0 . Then there must exist a set of

positive probability measure in the space of $\{x_i\}_{i=1}^{\infty}$ such that $x_i \in \mathbb{R}^n$ and $\sum_{i=1}^{\infty} \|x_i\|_2^2 < \infty$.

derive

$$x_{i+1} = x_i + \Delta x_i \quad \text{for } i = 0, 1, \dots, n-1$$

$$.0 \leq [x, \dots, 1] = 0, [1+x]_2^x = [1+x]_2^x|_0 =$$

as mentioned you thought you'd avoid review.

$\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A and $\mu_1, \mu_2, \dots, \mu_n$ are the eigenvalues of B .

probability is the same as the unconditional probability which is zero by

6. The insurance shall not be voiding at any time, if the insured is not insured.

is 0 which is the same as saying $\text{vec}(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}) = 0$ But the

[illegible]

$$.0 = (\text{value at } \bar{z}) \text{ const}$$

Theorem 2.

Suppose $\underline{X}_1, \dots, \underline{X}_N$ are mutually independent and for every $\alpha (\alpha=1, \dots, N)$ and for every $(p-1)$ -dimensional hyperplane L in R^p , $\text{Prob}(\underline{x}_\alpha \in L) = 0$. Then $\text{Prob}(A \text{ is singular}) = 0$.

Proof:

$$\text{Note that} \quad \det A = N \det \left[\begin{array}{c|c} \sum_{\alpha=1}^N \underline{x}_\alpha \underline{x}'_\alpha & \underline{\bar{x}} \\ \hline \underline{\bar{x}}' & 1/N \end{array} \right] = N \det \left[\sum_{\alpha=1}^N \underline{y}_\alpha \underline{y}'_\alpha \right]$$

where $\underline{y}'_\alpha = (\underline{x}'_\alpha, 1/N)$. Hence

$$A \text{ is singular} \Leftrightarrow \sum_{\alpha=1}^N \underline{y}_\alpha \underline{y}'_\alpha \text{ is singular.}$$

Suppose A is singular with positive probability. Then $(\underline{y}_1, \dots, \underline{y}_N)$ is of rank $< (p+1)$ with positive probability. Hence (without loss of generality)

$$\det[\underline{y}_1, \dots, \underline{y}_{p+1}] = 0$$

with positive probability. Note that $p+1 \leq N$. Thus

$$\begin{aligned} (1) \quad & 0 < \text{Prob}\{\det(\underline{y}_1, \dots, \underline{y}_{p+1}) = 0\} \\ & = E\{\text{Prob}[\det(\underline{y}_1, \dots, \underline{y}_{p+1}) = 0 \mid \underline{y}_\alpha = \underline{y}_\alpha, \alpha = 1, \dots, p]\} \end{aligned}$$

where the last coordinate of \underline{y}_α is $1/N$ ($\alpha = 1, \dots, p$). But

$\det(\underline{y}_1, \dots, \underline{y}_{p+1}) = 0$ can be expressed as $\underline{b}'\underline{x}_{p+1} = a$ where \underline{b} and a are functions of $\underline{y}_1, \dots, \underline{y}_p$. The conditional probability in (1) is

$$(2) \quad \text{Prob}[\underline{b}'\underline{x}_{p+1} = a],$$

since \underline{x}_{p+1} is independent of $\underline{x}_1, \dots, \underline{x}_p$. (2) is zero unless $\underline{b} = \underline{0}$ and $a = 0$. Hence, in order that (1) holds, we must have

and $\varepsilon = 0$. Hence: in order that (1) holds, we must have

that \tilde{X}^{b+1} is independent of $\tilde{X}^1, \dots, \tilde{X}^b$ (5) is also true $\tilde{p} = 0$

$$(5) \quad \text{Evar}[\tilde{p}, \tilde{X}^{b+1} = \varepsilon] = 0.$$

is the condition of $\tilde{X}^1, \dots, \tilde{X}^b$. The corresponding independence is (1) is

$$\text{var}(\tilde{X}^1, \dots, \tilde{X}^{b+1}) = 0 \text{ can be expressed as } \tilde{p}, \tilde{X}^{b+1} = \varepsilon \text{ and } \tilde{p} \text{ and}$$

where the last condition of \tilde{X}^1 is \tilde{X}^1 ($\tilde{X}^1 = \tilde{X}^1, \dots, \tilde{X}^b$). For

$$= E\{\text{var}(\tilde{X}^1, \dots, \tilde{X}^{b+1}) = 0 | \tilde{X}^1 = \tilde{X}^1, \dots, \tilde{X}^b = \tilde{X}^b\}$$

$$(7) \quad 0 < \text{var}(\tilde{X}^1, \dots, \tilde{X}^{b+1}) = 0$$

then becomes independent. Now that $b+1 \leq n$. Thus.

$$\text{var}(\tilde{X}^1, \dots, \tilde{X}^{b+1}) = 0$$

of rank $< (b+1)$ then becomes independent. Hence (in order that the condition,

holds, \tilde{X}^1 is independent then becomes independent. Thus $(\tilde{X}^1, \dots, \tilde{X}^b)$ is

$$\tilde{X}^1 \text{ is independent} \Leftrightarrow \begin{matrix} c=1 \\ \tilde{X}^1 \end{matrix} \begin{matrix} a, a \\ \tilde{X}^1 \end{matrix} \text{ is independent.}$$

where $\tilde{X}^1 = (\tilde{X}^1, \dots, \tilde{X}^b)$. Hence

$$\text{var } \tilde{X}^1 = E \text{var} \left[\begin{array}{c|c} \tilde{X}^1 & \tilde{X}^1 \\ \hline \tilde{X}^1 & \tilde{X}^1 \end{array} \right] = E \text{var} \left[\begin{array}{c|c} \tilde{X}^1 & \tilde{X}^1 \\ \hline \tilde{X}^1 & \tilde{X}^1 \end{array} \right]$$

Now that

Thus:

$$\text{var}(\tilde{X}^1 \in \tilde{X}^1) = 0. \text{ Then } \text{var}(\tilde{X}^1 \text{ is independent}) = 0.$$

$(c=1, \dots, n)$ and for each $(b+1)$ -dimensional subspace \tilde{X}^1 in \tilde{X}^1

holds $\tilde{X}^1, \dots, \tilde{X}^b$ are independent and for each

Lemma 5.

$$(3) \quad \text{Prob}[\det(\underline{X}_1, \dots, \underline{X}_p) = 0] > 0.$$

Now it follows from the lemma that (3) is false.

Theorem 3.

Let $\underline{X}_1, \dots, \underline{X}_N$ be i.i.d. $p \times 1$ random vectors ($p \leq N - 1$). Then $\text{Prob}(A \text{ is singular}) = 0$ iff for every $(p-1)$ -dimensional hyperplane L in R^p , $\text{Prob}(\underline{X}_\alpha \in L) = 0$.

Proof:

Use Theorems 1 and 2.

Further Results.

Let us now consider the question of nonsingularity (with probability 1) of $\bar{X} M \bar{X}'$ where \bar{X} is defined as before and M is an $N \times N$ p.s.d. matrix.

Let $\text{rank}(M) = r$. Then M can be expressed as

$$M = L_1 \Lambda L_1'$$

where $L_1 : N \times r$, $L_1' L_1 = I_r$ and Λ is a diagonal matrix with all diagonal elements positive. Note now

$$\begin{aligned} \text{rank}(\bar{X} M \bar{X}') &= \text{rank}(\bar{X} L_1 \Lambda^{1/2}) = \text{rank}(\bar{X} L_1) \\ &= \text{rank}(\bar{X} L_1 L_1' \bar{X}') \\ &= \text{rank}[\bar{X} (I_N - L_1 L_1') \bar{X}'] \end{aligned}$$

where $L_2 : N \times (N-r)$, $L_2' L_2 = I_{N-r}$, $L_2' L_1 = 0$. Thus

$$\det[\bar{X} (I_N - L_2 L_2') \bar{X}'] = \det \left[\begin{pmatrix} \bar{X} \\ L_2' \end{pmatrix} \begin{pmatrix} \bar{X}' & L_2' \end{pmatrix} \right].$$

Hence $\bar{X} M \bar{X}'$ is singular $\Leftrightarrow \text{rank} \begin{pmatrix} \bar{X} \\ L_2' \end{pmatrix} < p + (N-r)$. It is clear that

if $p > r$, $\bar{X} M \bar{X}'$ is singular. We shall assume now $p \leq r$.

$$(3) \quad \text{Prob}(\lambda_1, \dots, \lambda_p) = 0 \text{ if } 0 < \lambda_1 \leq \dots \leq \lambda_p \leq 1.$$

Now it follows from the lemma that (3) is false.

Theorem 2.

Let $\lambda_1, \dots, \lambda_p$ be a set of $p \times p$ random vectors (p.d.f. $\lambda_1, \dots, \lambda_p$). Then
 The λ_i are singular $\Leftrightarrow 0 = \lambda_i$ for every i (p.d.f. $\lambda_1, \dots, \lambda_p$)
 in \mathbb{R}^p , $\text{Prob}(\lambda_1, \dots, \lambda_p) = 0$.

Proof:

Use Theorems 1 and 2.

Further Results.

Let us now consider the question of non-singularity (with probability 1) of $\bar{X} M \bar{X}'$, where \bar{X} is defined as before and M is an $n \times n$ p.d.f. matrix.

Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of M . Then M can be expressed as

$$M = \sum_{i=1}^n \lambda_i A_i A_i'$$

where $A_i: n \times 1$, $A_i A_i' = I$ and A_i is a diagonal matrix with all diagonal elements positive. Note now

$$\text{rank}(\bar{X} M \bar{X}') = \text{rank}(\bar{X} \bar{X}') = \text{rank}(\bar{X})$$

$$= \text{rank}(\bar{X} \bar{X}') = \text{rank}(\bar{X})$$

$$= \text{rank}(\bar{X} \bar{X}') = \text{rank}(\bar{X})$$

where $\bar{X}: n \times (n-p)$, $\bar{X} \bar{X}' = I - \sum_{i=1}^p A_i A_i'$, $\lambda_i = 0$. Thus

$$\text{rank}(\bar{X} \bar{X}') = \text{rank}(\bar{X}) = \text{rank}(\bar{X} \bar{X}') = \text{rank}(\bar{X})$$

Hence $\bar{X} M \bar{X}'$ is singular $\Leftrightarrow \text{rank}(\bar{X}) < n-p$. It is clear that

if $p < n$, $\bar{X} M \bar{X}'$ is singular. We shall assume now $p \leq n$.

Theorem 4.

Suppose there exist $(p-1)$ -dimensional hyperplanes L_1, \dots, L_N where $L_\alpha : \underline{b}'\underline{x} = a_\alpha$ and $(a_1, \dots, a_N) \perp C(M)$, where $C(M)$ is the space spanned by the column vectors of M , such that $P[\underline{X}_\alpha \in L_\alpha, \alpha = 1, \dots, N] > 0$, where $\underline{X}_1, \dots, \underline{X}_N$ are $p \times 1$ random vectors. Then

$$P(\underline{\bar{X}} M \underline{\bar{X}}' \text{ is singular}) > 0.$$

Proof:

$$\underline{X}_\alpha \in L_\alpha, \alpha = 1, \dots, N \Rightarrow$$

$$\begin{aligned} \underline{b}' \underline{\bar{X}} M \underline{\bar{X}}' \underline{b} &= (a_1, \dots, a_N) M (a_1, \dots, a_N)' \\ &= 0 \end{aligned}$$

$$\Rightarrow \underline{\bar{X}} M \underline{\bar{X}}' \text{ is singular.}$$

Theorem 5.

Let $\underline{X}_1, \dots, \underline{X}_N$ be mutually independent $p \times 1$ random vectors. Suppose for every $\alpha (\alpha = 1, \dots, N)$ and for every $(p-1)$ -dimensional hyperplane L in R^p , $\text{Prob}(\underline{X}_\alpha \in L) = 0$. Then, $\text{Prob}(\underline{\bar{X}} M \underline{\bar{X}}' \text{ is singular}) = 0$.

Proof:

It was shown that

$$\underline{\bar{X}} M \underline{\bar{X}}' \text{ is singular} \Leftrightarrow \text{rank} \begin{pmatrix} \underline{\bar{X}} \\ L_2' \end{pmatrix} < p + N - r.$$

Without loss of generality, assume

$$L_2' = (Q_1 \quad Q_2 \quad Q_3)$$

where $Q_1 : (N-r) \times (N-r)$ is of rank $N - r$, $Q_2 : (N-r) \times p$. Similarly, partition as

$$\begin{pmatrix} \underline{\bar{X}} \\ L_2' \end{pmatrix} = \begin{pmatrix} \underline{\bar{X}}^{(1)} & \underline{\bar{X}}^{(2)} & \underline{\bar{X}}^{(3)} \\ Q_1 & Q_2 & Q_3 \end{pmatrix}$$

$$\begin{pmatrix} \bar{r}_1^S \\ \bar{r}_2^S \end{pmatrix} = \begin{pmatrix} \bar{c}^I & \bar{c}^S & \bar{c}^B \end{pmatrix} \begin{pmatrix} \bar{x}(r) \\ \bar{x}(s) \\ \bar{x}(b) \end{pmatrix}$$

беремому \bar{x}

матрица $\bar{c}^I : (n-1) \times (n-1)$ та же как и $\bar{c}^S : (n-1) \times 1$. следовательно:

$$\bar{r}_1^S = (\bar{c}^I \quad \bar{c}^S \quad \bar{c}^B)$$

матрица \bar{r}_2^S та же как и \bar{r}_1^S .

$$\bar{x} \in \bar{X}, \text{ та матрица } \bar{c}^I \text{ та же как } \bar{c}^S, \bar{c}^B < 1 + n - 1.$$

та же матрица

матрица:

матрица \bar{r}_1^S та же как \bar{r}_2^S : $\bar{r}_1^S \in \bar{r}_2^S$. $\bar{r}_1^S \in \bar{r}_2^S$. $\bar{r}_1^S \in \bar{r}_2^S$. $\bar{r}_1^S \in \bar{r}_2^S$.

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та же $\bar{r}_1^S, \dots, \bar{r}_n^S$ та же как \bar{r}_2^S . $\bar{r}_1^S \in \bar{r}_2^S$. $\bar{r}_1^S \in \bar{r}_2^S$. $\bar{r}_1^S \in \bar{r}_2^S$. $\bar{r}_1^S \in \bar{r}_2^S$.

матрица \bar{r}_1^S .

$$\Rightarrow \bar{x} \in \bar{X}, \text{ та матрица } \bar{c}^I \text{ та же как } \bar{c}^S, \bar{c}^B < 1 + n - 1.$$

$$= 0$$

$$P, \bar{x} \in \bar{X}, P = (\bar{c}^I, \dots, \bar{c}^B) \bar{r}_1^S, \dots, \bar{r}_n^S,$$

$$P, \bar{x} \in \bar{X}, P = (\bar{c}^I, \dots, \bar{c}^B) \bar{r}_1^S, \dots, \bar{r}_n^S,$$

матрица:

$$\bar{x} \in \bar{X}, \text{ та матрица } \bar{c}^I \text{ та же как } \bar{c}^S, \bar{c}^B < 1 + n - 1.$$

матрица $\bar{r}_1^S, \dots, \bar{r}_n^S$ та же как \bar{r}_2^S . $\bar{r}_1^S \in \bar{r}_2^S$. $\bar{r}_1^S \in \bar{r}_2^S$. $\bar{r}_1^S \in \bar{r}_2^S$. $\bar{r}_1^S \in \bar{r}_2^S$.

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матрица \bar{r}_1^S .

$$\text{rank}\left(\frac{\bar{X}}{L_2'}\right) < p + N - r \Rightarrow \det\begin{bmatrix} \bar{X}^{(1)} & \bar{X}^{(2)} \\ Q_1 & Q_2 \end{bmatrix} = 0$$

$$\Leftrightarrow \det[\bar{X}^{(2)} - \bar{X}^{(1)} Q_1^{-1} Q_2] = 0.$$

Define

$$Z = (Z_1, \dots, Z_p) = \bar{X}^{(2)} - \bar{X}^{(1)} Q_1^{-1} Q_2.$$

Given $\bar{X}^{(1)} = x^{(1)}$, Z_1, \dots, Z_p are mutually independent (conditionally). Moreover, given $\bar{X}^{(1)} = x^{(1)}$, probability that any Z_α ($\alpha = 1, \dots, p$) lies in any $(p-1)$ -dimensional hyperplane in R^p is 0. Using the lemma, it is easy to show now

$$\text{Prob}\{\det[\bar{X}^{(2)} - \bar{X}^{(1)} Q_1^{-1} Q_2] = 0 | \bar{X}^{(1)} = x^{(1)}\} = 0.$$

Hence

$$\text{Prob}(\bar{X} M \bar{X}' \text{ is singular}) = 0.$$

This result directly follows from the lemma; an alternative proof is given to show a different approach.

Comments.

(1) If $e_N \perp C(M)$ and X_1, \dots, X_N are i.i.d., then $\text{Prob}(\bar{X} M \bar{X}' \text{ is singular}) = 0$ iff for every $(p-1)$ -dimensional hyperplane L in R^p , $\text{Prob}(X_\alpha \in L) = 0$.

(2) If $p = 1$, and $\bar{X}' = (X_1, \dots, X_N)$ are mutually independent random variables, then $\text{Prob}(\bar{X}' M \bar{X} \text{ is singular}) = 0$ iff $\text{Prob}(\bar{X} \perp C(M)) = 0$.

The above results can be shown easily.

аналогично до более того.

ре аналогично до более того. $\text{Значит } (\bar{X} \text{ и } \bar{Y}, \text{ не связывая}) \text{ до ре } 0 \text{ для для связывая}$

(2) $\text{It seems that the relation of the solution in direction } \beta \text{ holds}$
 $\text{для более связывая сн ре связывая связывая.}$

аналогично до более того. $\text{Значит } (\bar{X}, \bar{Y} \text{ и } \bar{Z}, \text{ не связывая}) = 0 \text{ для } \text{Значит } (\bar{X} \text{ и } \bar{Y}) = 0.$

(1) $\text{It is } \beta = \gamma \text{ and } \bar{X} = (X^1, \dots, X^N) \text{ and analogical representation}$
 $\text{Значит } (\bar{X} \text{ и } \bar{Y}) = 0.$

аналогично до более того. $\text{Значит } (\beta - \gamma) \text{-аналогично до более того } \gamma \text{ и } \beta.$

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Значит

$$\text{Значит } (\bar{X}(\beta) - \bar{X}(\gamma) \cdot \beta) = 0 \text{ для } (\bar{X}(\gamma) = \beta(\gamma)) = 0.$$

Значит до более того.

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$$\beta = (\beta^1, \dots, \beta^N) = \bar{X}(\beta) - \bar{X}(\gamma) \cdot \beta.$$

Значит

$$\text{Значит } (\bar{X}(\beta) - \bar{X}(\gamma) \cdot \beta) = 0.$$

$$\text{Значит } (\bar{X}(\beta) - \bar{X}(\gamma) \cdot \beta) = 0.$$

This note was originally written primarily for class-lectures.
Dr. M. D. Perlman brought the reference in [1] to my attention and
mentioned the problem of studying $\bar{X} M \bar{X}'$.

Reference.

- [1] Dykstra, R. L. (1970). Establishing the positive-definiteness of
the sample covariance matrix. Ann. Math. Statist. 41 2153-2154.

the the entire collection known. The first series is 11 1120-1129.

[1] Dikens, E. P. (1840). Descriptions of the botanical-geographical of
reference.

mentioned the location of the site \bar{Y} in \bar{X} .

Dr. H. D. Berman points to the attention the reference in [1] and

the note has originally written Berman for manuscript-reference.